Linear Algebra & Geometry

LECTURE 13

- Linear mappings
- Nullity and rank
- Matrices of linear mappings

Linear Mappings

Definition. (reminder)

Let V and W be vector spaces over a field $\mathbb{F}$. A function $\varphi: V \to W$ is called a *linear mapping* iff

(a) $(\forall u, v \in V)\ \varphi(u+v) = \varphi(u) + \varphi(v)$ (additivity),

(b) $(\forall v \in V)(\forall p \in \mathbb{F})\ \varphi(pv) = p\varphi(v)$ (homogeneity) .

Example.

For every $n \times k$ matrix A the function $f_{\mathrm{A}}: \mathbb{F}^{k\times p} \to \mathbb{F}^{n\times p}$ defined as $f_A(X) = AX$ is a linear mapping.

Additivity follows from distributivity of matrix multiplication over addition.

Homogeneity is obvious: for every i and j, $f_A(\lambda X)(i,j) = A(\lambda X)(i,j) = \sum_{s=1}^{n} A(i,s)\lambda X(s,j) = \lambda \sum_{s=1}^{n} A(i,s)X(s,j) = \lambda\big(AX(i,j)\big) = \lambda f_A(X)(i,j)$.

Rank and nullity

Definition. Let $\varphi: V \to W$ be a linear mapping. The *image* of φ is the set $\text{im}\,\varphi = \varphi(V)$ and the *kernel* of φ is the set

$$\ker\varphi = \{v \in V: \varphi(v) = \Theta_W\}$$

Proposition. $\text{im}\varphi$ is a subspace of W and $\ker\varphi$ is a subspace of V.

Proof. Let $w_1, w_2 \in im\varphi$. There exist $v_1, v_2 \in V$ such that $\varphi(v_i) = w_i, i = 1,2$. Then $pw_1 + qw_2 = p\varphi(v_1) + q\varphi(v_2) = \varphi(pv_1 + qv_2) \in im\varphi$, hence $im\varphi$ is a subspace of W.

For every $v_1, v_2 \in \ker\varphi$, $\varphi(av_1 + bv_2) = a\varphi(v_1) + b\varphi(v_2) = a\Theta + b\Theta = \Theta$ hence, $\ker\varphi$is a subspace of V. QED

Definition. $rank(\varphi) = \dim(\text{im}\varphi), nullity(\varphi) = \dim(\ker\varphi)$.

Example. Let $V = W = \mathbb{R}_n[x]$ and $\varphi(f(x)) = f\,'(x)$. Then $\ker\varphi = \mathbb{R}_0[x]$, the space of all constant polynomials and $\text{im}\varphi$= $\mathbb{R}_{n-1}[x]$. Hence, $rank(\varphi) = n$ and $nullity(\varphi) = 1$.

Proposition. For every set $\{v_1, v_2, \dots, v_n\} \subseteq V$ and for every linear mapping $\varphi: V \to W$,
$\varphi(span(v_1, v_2, \dots, v_n)) = span(\varphi(v_1), \varphi(v_2), \dots, \varphi(v_n))$.
Proof. (Obvious)

Example. For every $n \times k$ matrix A, $f(X) = AX$ is a linear mapping, $f: \mathbb{F}^k \to \mathbb{F}^n$. ker$f$ is the solution set (space, really) for $AX = \Theta$ hence, *nullity* of f is the dimension of the solution space. On the other hand, denoting by $e_1, e_2, \dots, e_k$ the unit vectors of $\mathbb{F}^k$ we obtain that im f is spanned by $\{f(e_1), f(e_2), \dots, f(e_k)\} = \{Ae_1, \dots, Ae_k\}$ i.e., by the columns of A. Hence, the rank of f is the number of linearly independent columns of A which is the same as rank of A.

Theorem.
For every linear mapping $\varphi: V \to W$

$$rank(\varphi) + nullity(\varphi) = dimV$$

Proof. Let $\{v_1, v_2, \dots, v_n\}$ be a basis for kerφ. There exist vectors $w_1, w_2, \dots, w_k$ such that $\{v_1, v_2, \dots, v_n, w_1, w_2, \dots, w_k\}$ is a basis for V. It is enough to show that $S = \{\varphi(w_1), \varphi(w_2), \dots, \varphi(w_k)\}$ is a basis for im φ. Since $span(S) =$ im φ, it is enough to show that S is linearly independent. Let $\sum_{i=1}^{k} a_i \varphi(w_i) = \Theta_W$. By linearity of φ, $\Theta_W = \sum_{i=1}^{k} a_i \varphi(w_i) = \varphi(\sum_{i=1}^{k} a_i w_i)$ i.e., $\sum_{i=1}^{k} a_i w_i \in$ kerφ so, for some scalars $b_1, b_2, \dots, b_n$ we have $\sum_{i=1}^{k} a_i w_i = \sum_{j=1}^{n} b_j v_j$. This implies $\sum_{i=1}^{k} a_i w_i - \sum_{j=1}^{n} b_j v_j = \Theta_V$ hence, all a_i and b_j are zeroes. QED

Corollary.
If A is an $n \times k$ matrix then the dimension of the solution space of the homogeneous system of equations $AX = \Theta$ is $k - rank(A)$.

Note.
Every n-dimensional vector space V over a field $\mathbb{F}$ is *isomorphic* with $\mathbb{F}^n$.
Proof. Let $R = \{v_1, v_2, \dots, v_n\}$ be a basis of V. The function $\Phi(x) = (a_1, a_2, \dots, a_n)$, where $a_1, a_2, \dots, a_n$ are unique scalars such that $x = \sum_{s=1}^{n} a_s v_s$ maps V into $\mathbb{F}^n$ is an isomorphism. QED

The vector $(a_1, a_2, \dots, a_n)$ is called the *coordinate vector* of x with respect to R and is often denoted by $[x]_R$.
From now on we will use $\mathbb{F}^n$, rather than general symbol V of a vector space.

Definition. Let $\varphi: \mathbb{F}^k \to \mathbb{F}^n$ be a linear mapping and let $R = \{v_1, v_2, \ldots, v_k\}$ and $S = \{w_1, w_2, \ldots, w_n\}$ be bases for $\mathbb{F}^k$ and $\mathbb{F}^n$, respectively. For each v_i there exist unique scalars $a_{1,i}, a_{2,i}, \ldots,$ $a_{n,i}$ such that $\varphi(v_i) = a_{1,i}w_1 + a_{2,i}w_2 + \ldots + a_{n,i}w_n = \sum_{s=1}^{n} a_{s,i}w_s$. The $k \times n$ matrix $M_S^R(\varphi) = [a_{i,j}]$ is called the matrix of φ in bases R and S.

Remark. In other words, the i-th column of the matrix $M_S^R(\varphi)$ is equal to $[\varphi(v_i)]_S$, the coordinate vector of $\varphi(v_i)$ in the basis S.

Remark. The matrix of a linear mapping φ depends on the choice of the bases R and S, but the size of the matrix depends only on the dimensions of the domain and the range of φ.

It may happen that two different linear mappings may have the same matrix but with respect to two pairs of bases.

Example.

Let $\varphi: \mathbb{R}^3 \to \mathbb{R}^2$, $\varphi(x, y, z) = (x + 2y - 2z, 3x - y + 2z)$. Find the matrix for φ in bases $R = \{(0,1,1), (1,0,1), (1,1,0)\}$ and $S = \{(1,1), (1,0)\}$.

We must calculate the values of φ on vectors from R and represent them as linear combinations of vectors from S.

$\varphi(0,1,1) = (0,1) = 1(1,1) + (-1)(1,0)$,

$\varphi(1,0,1) = (-1,5) = 5(1,1) + (-6)(1,0)$,

$\varphi(1,1,0) = (3,2) = 2(1,1) + 1(1,0)$.

Finally, we form the matrix placing the coefficients of the linear combinations in consecutive columns, $M_S^R(\varphi) = \begin{bmatrix} 1 & 5 & 2 \\ -1 & -6 & 1 \end{bmatrix}$.

Theorem.
Let $\varphi\colon \mathbb{F}^k \to \mathbb{F}^n$ be a linear mapping, let $R = \{v_1, v_2, \dots, v_k\}$, $S = \{w_1, w_2, \dots, w_n\}$ be bases for $\mathbb{F}^k$ and $\mathbb{F}^n$, respectively, and let A be an $n \times k$ matrix. Then:

$$\left((\forall x \in \mathbb{F}^k)[\varphi(x)]_S = A[x]_R\right) \Leftrightarrow A = M_S^R(\varphi).$$

Proof. (⇐) Let $M_S^R(\varphi) = A = [a_{i,j}]$ and $[x]_R = [x_1, x_2, \dots, x_k]$ i.e., $x = \sum_{i=1}^k x_i v_i$. Then, $\varphi(x) = \varphi\left(\sum_{i=1}^k x_i v_i\right) = \sum_{i=1}^k x_i \varphi(v_i) = \sum_{i=1}^k x_i \sum_{s=1}^n a_{s,i} w_s = \sum_{i=1}^k \sum_{s=1}^n x_i (a_{s,i} w_s) = \sum_{s=1}^n (\sum_{i=1}^k x_i a_{s,i}) w_s$

(**comprehension:** why can we change the order of summation?)

which means $[\varphi(x)]_S = (\sum_{i=1}^k x_i a_{1,i}, \sum_{i=1}^k x_i a_{2,i}, \dots, \sum_{i=1}^k x_i a_{k,i})$.
$M_S^R(\varphi)[x]_R = A\,[x]_R = (\sum_{i=1}^k a_{1,i}\, x_i, \sum_{i=1}^k a_{2,i} x_i, \dots, \sum_{i=1}^k a_{k,i}\, x_i)$,
(except that the vectors $[\varphi(x)]_S$ and $M_S^R(\varphi)[x]_R$ should be written as columns).
(⇒) Replacing x with v_i one gets i-th column of $M_S^R(\varphi)$. QED

Fact.

All linear mappings are functions of the form $\varphi(X) = AX$.

Theorem.

Let $\varphi: \mathbb{F}^k \to \mathbb{F}^n$ and $\psi: \mathbb{F}^n \to \mathbb{F}^p$ be linear maps and let R, S and T be bases for $\mathbb{F}^k, \mathbb{F}^n$ and $\mathbb{F}^p$, respectively. Then

$$M_T^R(\psi \circ \varphi) = M_T^S(\psi) M_S^R(\varphi).$$

Proof.

For every $x \in \mathbb{F}^k$, $M_T^R(\psi \circ \varphi)[x]_R = [(\psi \circ \varphi)(x)]_T = \left[\psi\big(\varphi(x)\big)\right]_T = M_T^S(\psi)[\varphi(x)]_S = M_T^S(\psi)\big(M_S^R(\varphi)[x]_R\big) = \Big(M_T^S(\psi) M_S^R(\varphi)\Big)[x]_R$.

From the *if* part of the previous theorem we get

$M_T^R(\psi \circ \varphi) = M_T^S(\psi) M_S^R(\varphi)$. QED

Corollary.

Matrix multiplication is associative.

Linear operators

From now on, we will study linear maps which map a vector space into itself. They are called *linear operators*.

If φ is a linear operator and R is a basis then $M_R^R(\varphi)$ is called the matrix of φ in (with resp. to) R and is denoted by $M_R(\varphi)$.

Example.

Consider $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$, $\varphi(x, y) = (x + y, x - y)$. Find $M_S(\varphi)$ and $M_R(\varphi)$, where $S = \{(1,0), (0,1)\}$ and $R = \{(1,1), (2,1)\}$.

To find $M_R(\varphi)$:

$\varphi(1,1) = (2,0) = a(1,1) + b(2,1)$. Solving the system of equations $\begin{cases} a + 2b = 2 \\ a + b = 0 \end{cases}$ we get $a = -2, b = 2$.

$\varphi(2,1) = (3,1) = c(1,1) + d(2,1)$. Solving the system of equations $\begin{cases} c + 2d = 3 \\ c + d = 1 \end{cases}$ we get $c = -1, d = 2$. Hence, $M_R(\varphi) = \begin{bmatrix} -2 & -1 \\ 2 & 2 \end{bmatrix}$.

Obviously, $M_S(\varphi) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ (because $\varphi(1,0) = (1,1) = 1(1,0) + 1(0,1)$ and $\varphi(0,1) = (1, -1) = 1(1,0) + (-1)(0,1)$).